NOTE ON STOKES'S THEOREM IN CURVILINEAR CO-ORDINATES.

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THE expressions for the components of the curl of a vector point-function, when required in terms of orthogonal curvilinear co-ordinates, are usually obtained by direct transformation of their values in rectangular co-ordinates.

The proof of Stokes's Theorem, given in my Lectures on Electricity and Magnetism, due to Helmholtz, may be easily adapted to curvilinear co-ordinates so as to prove the theorem independently of rectangular co-ordinates.

Let P_1, P_2, P_3, be the projections of a vector P on the varying directions of the co-ordinate axes at any point. Let the projections on the same axes of the arc ds of a curve connecting the points A and B be ds_1, ds_2, ds_3. The theorem concerns the line integral of the resolved component of the vector along the given curve.

$$\begin{aligned} I &= \int_A^B P \cos(P, ds)\, ds \\ &= \int_A^B P_1\, ds_1 + P_2\, ds_2 + P_3\, ds_3. \end{aligned}$$

But in terms of the curvilinear co-ordinates ρ_1, ρ_2, ρ_3, we have

$$ds_1 = \frac{d\rho_1}{h_1}, \quad ds_2 = \frac{d\rho_2}{h_2}, \quad ds_3 = \frac{d\rho_3}{h_3},$$

where

$$h_s^2 = \left(\frac{\delta \rho_s}{\delta x}\right)^2 + \left(\frac{\delta \rho_s}{\delta y}\right)^2 + \left(\frac{\delta \rho_s}{\delta z}\right)^2. \qquad s = 1, 2, 3.$$

Let us now make an infinitesmal transformation of the curve, so that the transformed curve shall lie on a given surface containing A and B,

and shall itself pass through those points. Then the change in the integral due to changes in the co-ordinates $\delta\rho_1$, $\delta\rho_2$, $\delta\rho_3$, is,

$$\delta I = \delta \int \frac{P_1}{h_1} d\rho_1 + \frac{P_2}{h_2} d\rho_2 + \frac{P_3}{h_3} d\rho_3$$

$$= \int \delta\left(\frac{P_1}{h_1}\right) d\rho_1 + \delta\left(\frac{P_2}{h_2}\right) d\rho_2 + \delta\left(\frac{P_3}{h_3}\right) d\rho_3 + \frac{P_1}{h_1} d\,\delta\rho_1 + \frac{P_2}{h_2} d\,\delta\rho_2 + \frac{P_3}{h_3} d\,\delta\rho_3.$$

The last three terms may be integrated by parts, giving

$$\int_A^B \frac{P_s}{h_s} d\,\delta\rho_s = \frac{P_s}{h_s}\,\delta\rho_s \Big/_A^B - \int_A^B \delta\rho_s\, d\left(\frac{P_s}{h_s}\right),$$

and the integrated part vanishing at the limits,

$$\delta I = \int \delta\left(\frac{P_1}{h_1}\right) d\rho_1 + \delta\left(\frac{P_2}{h_2}\right) d\rho_2 + \delta\left(\frac{P_3}{h_3}\right) d\rho_3 - \delta\rho_1\, d\left(\frac{P_1}{h_1}\right) - \delta\rho_2\, d\left(\frac{P_2}{h_2}\right) - \delta\rho_3\, d\left(\frac{P_3}{h_3}\right).$$

Performing the operations denoted by δ and d, and collecting the terms which do not cancel,

$$\begin{aligned}\delta I = \int \Bigg[&\left(\delta\rho_2\, d\rho_3 - \delta\rho_3\, d\rho_2\right) \left\{\frac{\delta}{\delta\rho_2}\left(\frac{P_3}{h_3}\right) - \frac{\delta}{\delta\rho_3}\left(\frac{P_2}{h_2}\right)\right\} \\ + &\left(\delta\rho_3\, d\rho_1 - \delta\rho_1\, d\rho_3\right) \left\{\frac{\delta}{\delta\rho_3}\left(\frac{P_1}{h_1}\right) - \frac{\delta}{\delta\rho_1}\left(\frac{P_3}{h_3}\right)\right\} \\ + &\left(\delta\rho_1\, d\rho_2 - \delta\rho_2\, d\rho_1\right) \left\{\frac{\delta}{\delta\rho_1}\left(\frac{P_2}{h_2}\right) - \frac{\delta}{\delta\rho_2}\left(\frac{P_1}{h_1}\right)\right\}.\end{aligned}$$

Now the changes $\delta\rho_s$, $d\rho_s$, in the co-ordinates correspond to distances $\frac{\delta\rho_s}{h_s}$, $\frac{d\rho_s}{h_s}$, measured along the co-ordinate lines, and the determinant of these distances,

$$\frac{1}{h_2 h_3}\left(\delta\rho_2\, d\rho_3 - \delta\rho_3\, d\rho_2\right),$$

is equal to the area of the projection on the surface ρ_1 of the infinitesimal parallelogram swept over by the arc ds during the transformation. Calling this area dS, and its normal n, we have

$$\frac{1}{h_2 h_3} (\delta \rho_2 \, d\rho_3 - \delta \rho_3 \, d\rho_2) = \cos (nn_1) \, dS,$$

$$\delta \rho_2 \, d\rho_3 - \delta \rho_3 \, d\rho_2 = h_2 h_3 \cos (nn_1) \, dS.$$

Now, repeating the transformation so that the original curve 1 passes into a second given curve 2, the total change is represented by the surface integral over the surface lying between the curves,

$$\int \delta I = I_2 - I_1 = \iint \Big[h_2 h_3 \left\{ \frac{\delta}{\delta \rho_2} \left(\frac{P_3}{h_3} \right) - \frac{\delta}{\delta \rho_3} \left(\frac{P_2}{h_2} \right) \right\} \cos (nn_1)$$
$$+ h_3 h_1 \left\{ \frac{\delta}{\delta \rho_3} \left(\frac{P_1}{h_1} \right) - \frac{\delta}{\delta \rho_1} \left(\frac{P_3}{h_3} \right) \right\} \cos (nn_2)$$
$$+ h_1 h_2 \left\{ \frac{\delta}{\delta \rho_1} \left(\frac{P_2}{h_2} \right) - \frac{\delta}{\delta \rho_2} \left(\frac{P_1}{h_1} \right) \right\} \cos (nn_3) \Big] \, dS.$$

But the difference of the line integrals $I_2 - I_1$ is the line integral around the closed contour 12, so that we have the line integral of the tangential component of the vector P around the closed contour proved equal to the surface integral over a surface bounded by the contour of the normal component of a vector Ω whose components are

$$\omega_1 = h_2 h_3 \left\{ \frac{\delta}{\delta \rho_2} \left(\frac{P_3}{h_3} \right) - \frac{\delta}{\delta \rho_3} \left(\frac{P_2}{h_2} \right) \right\}$$

$$\omega_2 = h_3 h_1 \left\{ \frac{\delta}{\delta \rho_3} \left(\frac{P_1}{h_1} \right) - \frac{\delta}{\delta \rho_1} \left(\frac{P_3}{h_3} \right) \right\}$$

$$\omega_3 = h_1 h_2 \left\{ \frac{\delta}{\delta \rho_1} \left(\frac{P_2}{h_2} \right) - \frac{\delta}{\delta \rho_2} \left(\frac{P_1}{h_1} \right) \right\}.$$

The vector Ω is called the curl of P.